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Casimir energy and geometry: beyond the proximity force approximation

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Abstract

We review the relation between the Casimir effect and geometry, emphasizing deviations from the commonly used proximity force approximation (PFA). We use, to this aim, the scattering formalism which is nowadays the best tool available for accurate and reliable theory–experiment comparisons. We first recall the main lines of this formalism when the mirrors can be considered to obey specular reflection. We then discuss the more general case where non-planar mirrors give rise to non-specular reflection with wavevectors and field polarizations mixed. The general formalism has already been fruitfully used for evaluating the effect of roughness on the Casimir force as well as the lateral Casimir force or Casimir torque appearing between corrugated surfaces. In this paper, we focus our attention to the case of the lateral force which should make possible in the future an experimental demonstration of the nontrivial (i.e. beyond PFA) interplay of the geometry and Casimir effect.

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1. Introduction

The Casimir force [1] is a remarkable prediction of quantum field theory. As the most easily accessible effect of vacuum fluctuations in the macroscopic world, it deserves careful experimental tests [2–4]. After tests, which confirmed its existence and main properties [5], experiments have been largely improved by technological achievements mastered over the last decade [6–13]. Meanwhile, it was realized that the Casimir force, a dominant force at micron or sub-micron distances, was clearly an important aspect of the study of micro- and nano-oscillators (MEMS, NEMS) [14, 15].

These recent advances have been reviewed in a number of papers, for example [16–18], and in a special issue of the *New Journal of Physics* [19]. In the following paragraphs, we emphasize

arguments which plead for careful comparisons between experimental measurements and theoretical predictions of the Casimir force [20].

1.1. Why testing the Casimir force?

A precise knowledge of the Casimir force is a key point for the tests of gravity at sub-millimeter ranges [21–23]. A strong constraint has been obtained recently in the short range Cavendish-like experiments [24]. Should a hypothetical new force have a Yukawa-like form, its strength could not be larger than that of the gravity for Yukawa ranges larger than $56 \mu\text{m}$.

Tests performed at shorter ranges essentially amount to comparisons with the theory of Casimir force measurements. In other words, the looked for hypothetical new force would correspond to an observable given by the difference $F_{\text{exp}} - F_{\text{th}}$ between the experiment and theory. This implies that the theoretical prediction F_{th} and the experimental measurement F_{exp} have to be treated independently from each other and with the same accuracy and reliability requirements.

To sum up the argument, the fact that the Casimir force experiments could be a window on hypothetical deviations from standard physics forbids one to use the theory–experiment comparison as an argument for proving (or disproving) some specific experiment or theoretical model. In this context, it is important to use a theoretical formalism having the ability to take into account the significant differences between the real experimental conditions and the ideal situation studied by Casimir [4, 17, 20].

Casimir calculated the force between a pair of perfectly smooth, flat and parallel plates in the limit of zero temperature and perfect reflection. He found expressions for the force F_{Cas} and energy E_{Cas} which only depend on the distance L , the area A and two fundamental constants, the speed of light c and the Planck constant \hbar ,

$$F_{\text{Cas}} = \frac{\hbar c \pi^2 A}{240 L^4} = \frac{dE_{\text{Cas}}}{dL}, \quad E_{\text{Cas}} = -\frac{\hbar c \pi^2 A}{720 L^3}. \quad (1)$$

This universality property of the Casimir expression is related to the saturation of the optical response of the mirrors when they reflect 100% of the incoming light. However, no real mirror can be considered as a perfect reflector at all field frequencies. The most precise experiments are performed with metallic mirrors which are good reflectors only at frequencies smaller than their plasma frequency. It follows that the Casimir force can obey the Casimir expression only at distances L larger than the plasma wavelength λ_p .

As this effect of imperfect reflection is large, a precise knowledge of its frequency dependence is essential for obtaining an accurate theoretical prediction of the Casimir force [25]. This is also true for another correction to the ideal Casimir formula associated with temperature effect. For discussions of this effect, we refer to discussions in [26, 27] and the recent review [28]. We now focus our attention on the effects of geometry which are also important in this context.

1.2. Why testing the effects of geometry?

It has been repeatedly stated over the years that the connection between the Casimir effect and geometry should show a rich variety of sensitive dependences [30–32]. The basis for this statement is the important fact that the Casimir forces cannot be additive, except in the specific case of interaction between very dilute media.

Meanwhile, most experiments are performed between a plane and a sphere with the Casimir force in this geometry calculated using the proximity force approximation [33], though the latter amounts to a mere averaging over the distribution of local interplate distances. The

PFA is expected to be valid in the plane–sphere geometry when the radius R is much larger than the separation L [34, 35] and is used to analyze most of the present-day experiments. Results going beyond this approximation have been obtained recently [36–45]. Some of these theoretical models involve scalar fields reflected on perfect boundary conditions and can hardly be compared with experiments, but those dealing with electromagnetic fields have now to be used for comparisons with experimental results obtained in the plane–sphere geometry.

As the effects of geometry on the Casimir force open access to a rich and stimulating physics, it is also important to explore this domain through new dedicated experiments. Only a few experiments have been designed to this aim which use a specific geometry with periodic corrugations imprinted on metallic surfaces. In this case, the Casimir force contains a lateral component since the lateral translation symmetry is broken [46]. The lateral Casimir force is smaller than the normal one, but has nevertheless already been measured in experiments [47]. The results have been found to agree within a bar of $\pm 24\%$ with PFA calculations.

Calculations beyond the PFA have also been performed by using more elaborate theoretical methods. The lateral force has been evaluated for perfectly reflecting mirrors using a path-integral formulation in a perturbative [48] or non-perturbative approach [49]. As the experiments are performed at distances L not much larger than the plasma wavelength λ_p , it is essential to account for the optical properties of the metals [25, 26]. Below we will present results obtained for corrugated metallic mirrors in the limiting case where the corrugation can be treated as a small perturbation [50–52]. As expected, the PFA is found to be valid when the corrugated surfaces appear as nearly plane to the vacuum fields involved in the calculation of the Casimir energy, that is to say when the corrugation wavelength λ_C is larger than the other relevant length scales.

1.3. Outline of the paper

We review below the theory of the Casimir effect within scattering theory. We will in particular present the formula giving the (QED) theoretical prediction for the Casimir force between scatterers placed in vacuum, or more generally at thermodynamical equilibrium with $T \neq 0$. This formula has been written years ago for plane and parallel mirrors showing specular reflection [53]. It has been used to discuss in a qualitative manner the effect of reflection properties of the mirrors on the Casimir force [17, 25, 26, 54]. Its applicability domain has been enlarged up to the point where it is now capable of dealing with non-planar geometries with non-specular reflection mixing field polarizations and transverse wavevectors [20]. We will recall below the application of this method to the calculation of the lateral Casimir force between corrugated plates [51, 52].

Note that similar discussions have been devoted to the discussion of the effect of surface roughness on the Casimir force. This description is commonly given within the PFA [55] which cannot remain valid for arbitrary roughness wavelengths [56]. As the effect of roughness is only a small correction of the Casimir force, one can however hardly expect quantitative theory–experiment comparisons in this case. This is why we will not discuss it below. Other applications have also been presented for the Casimir torque appearing between misaligned corrugation plates [57] and for the Casimir–Polder force between an atom or a cloud of atoms (BEC) and a corrugated metallic plate [58].

2. Specular scattering

We first consider the geometry with perfectly plane and parallel mirrors aligned along the directions x and y . As the configuration obeys a symmetry with respect to time translation as well as lateral space translations (along the directions x and y), the frequency ω , transverse

vector $\mathbf{k} \equiv (k_x, k_y)$ and polarization $p = \text{TE, TM}$ are preserved by the scattering processes which couple field modes having the same values for the preserved quantum numbers but a different sign for the longitudinal wavevector k_z . The two mirrors $j = 1, 2$ are described by the reflection and transmission amplitudes which depend on frequency, incidence angle $\theta = \arccos(ck_z/\omega)$ and polarization p .

2.1. Scattering formulae

The important result is that the Casimir force can be written in terms of the reflection amplitudes r_j of the two mirrors, as seen from inside the Fabry-Perot cavity formed by the two mirrors [53]. In order to write this relation, we introduce two functions which characterize the optical response of the cavity to an input field (dependences with respect to ω , \mathbf{k} and p are omitted):

$$f = \frac{r_1 r_2 e^{2ik_z L}}{1 - r_1 r_2 e^{2ik_z L}}, \quad g = 1 + f + f^* = \frac{1 - |r_1 r_2 e^{2ik_z L}|^2}{|1 - r_1 r_2 e^{2ik_z L}|^2}, \quad (2)$$

where f is the closed-loop function describing the cavity (L is the length of the cavity) and, therefore, obeys analyticity properties. Meanwhile, g is the ratio of the energy inside the cavity to the energy outside the cavity, that is also the ratio of the spectral density inside the cavity to the spectral density outside the cavity for a given mode. Its expression is valid for lossy as well as lossless mirrors as was demonstrated with an increasing range of validity in [53, 54, 59]. For lossy mirrors, it accounts for the additional fluctuations accompanying losses inside the mirrors.

Assuming thermal equilibrium for the whole ‘cavity + fields’ system, we obtain the radiation pressure exerted by the field fluctuations upon the mirrors. This leads to the following expression of the Casimir force as the sum over all field modes m of this radiation pressure (m gathers the parameters ω , \mathbf{k} and p):

$$\begin{aligned} F &= \sum_m \left(\frac{1}{2} + \bar{n} \right) \hbar \omega \cos^2 \theta \{1 - g\} \\ &= - \sum_m \left(\frac{1}{2} + \bar{n} \right) \hbar \omega \cos^2 \theta \{f + f^*\}. \end{aligned} \quad (3)$$

Here, $(\frac{1}{2} + \bar{n}) \hbar \omega$ is the mean energy per mode at temperature T with \bar{n} the mean number of photons per mode ($\bar{n} = 0$ at $T = 0$, $\bar{n} > 0$ otherwise):

$$\frac{1}{2} + \bar{n} = \frac{1}{2} \coth \frac{\hbar \omega}{2k_B T}. \quad (4)$$

Meanwhile, $\cos^2 \theta$ is a projection factor appearing in the translation from energy density to pressure; finally, $\{1 - g\}$ represents the difference between pressures on the outer and inner sides of the mirrors, respectively. Equation (3) contains the contribution of ordinary modes freely propagating outside and inside the cavity ($\omega > c|\mathbf{k}|$), which merely reflects the intuitive picture of a radiation pressure of field fluctuations on the mirrors [53]. But it also includes the contribution of evanescent waves ($\omega < c|\mathbf{k}|$) which propagate inside the mirrors with an incidence angle larger than the limit angle [54]. The properties of the latter are described through an analytical continuation of those of ordinary waves, using the well-defined analytic behavior of the function f .

Equation (3) can be equivalently written as a differential with respect to length of a free energy:

$$F = \frac{\partial \mathcal{F}}{\partial L}, \quad \mathcal{F} = \frac{\hbar c}{i} \sum_m \left(\frac{1}{2} + \bar{n} \right) \ln \frac{1 - r_1 r_2 e^{2ik_z L}}{1 - r_1^* r_2^* e^{-2ik_z L}}. \quad (5)$$

Using analyticity properties, as well as high-frequency transparency to neglect the contribution of large frequencies, both equations (3) and (5) can be transformed into integral over imaginary frequencies $\omega = i\xi$. We will write below more general forms of these relations, valid also for non-specular scattering.

2.2. The Lifshitz formula as a particular case

Equations (3) and (5) reproduce the Casimir formulae (1) in the limits of perfect reflection $r_1 r_2 \rightarrow 1$ and null temperature $T \rightarrow 0$. They are regular for any optical model of mirrors obeying causality and high-frequency transparency properties, without needing any further regularization. They can thus be used for calculating the Casimir force between arbitrary mirrors, as soon as the reflection amplitudes are specified. These amplitudes are commonly deduced from the microscopic models of mirrors, the simplest of which is the well-known Lifshitz model [60].

This model corresponds to plates having a large optical thickness, and characterized by a local dielectric function $\varepsilon(\omega)$. The reflection amplitudes are thus given by the Fresnel law written at the vacuum–bulk interface:

$$\begin{aligned} r^{\text{TE}} &= \frac{k_z - K_z}{k_z + K_z}, & r^{\text{TM}} &= \frac{K_z - \varepsilon k_z}{K_z + \varepsilon k_z}, \\ ck_z &= \sqrt{\omega^2 - c^2 k_z^2}, & cK_z &= \sqrt{\varepsilon \omega^2 - c^2 k_z^2}. \end{aligned} \quad (6)$$

Here, k_z and K_z correspond to the longitudinal wavevector in vacuum and in the bulk, respectively. Taken with equations (3) and (5) (possibly translated to the domain of imaginary frequencies), relations (6) reproduce the Lifshitz expression for the Casimir force [60]. The latter tend to the original Casimir expression in the limit $\varepsilon \rightarrow \infty$ which produces perfectly reflecting mirrors [61].

At this stage, several remarks are worth being emphasized:

- The expression of the force was not written in this manner by Lifshitz. To the best of our knowledge, Kats [62] was the first to note that the Lifshitz expression could be written in terms of the reflection amplitudes.
- The Lifshitz expression is valid for the cases for which it was derived. Its extension to more general situations can only be considered as valid after a careful examination of the derivation.
- In the most general case, the optical response of the bulk material cannot be described by a local dielectric function. In this case, the description in terms of reflection amplitudes, which necessarily differ from specific expressions (6), is still valid [20, 53, 54].

2.3. Description of real mirrors

In order to obtain a quantitative description of the effect of finite conductivity, we may in a first approach use expressions (6) with the dielectric function corresponding to the plasma model (ω_p the plasma frequency):

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad \varepsilon(i\xi) = 1 + \frac{\omega_p^2}{\xi^2}. \quad (7)$$

When performing these calculations, one recovers as expected the Casimir formula at large distances ($F \rightarrow F_{\text{Cas}}$ when $L \gg \lambda_p$). At distances smaller than λ_p in contrast, a significant reduction is obtained with the asymptotic law of variation read as

$$L \ll \lambda_p \quad \rightarrow \quad \frac{F}{F_{\text{Cas}}} \simeq 1.193 \frac{L}{\lambda_p}. \quad (8)$$

This can be understood as the result of the Coulomb interaction of surface plasmons at the two vacuum/metal interfaces [63, 64]. The generalization of this idea at arbitrary distances is more subtle since it involves a full electromagnetic treatment of the plasmon as well as ordinary photon modes [65].

The plasma model cannot provide a fully satisfactory description of the optical response of metals. A more realistic representation of the metals includes the description of the relaxation processes of conduction electrons as well as that of interband transitions. The reader is referred to [25] for a more detailed discussion³. The values of the complex index of refraction for different metals, measured through different optical techniques, are tabulated in several handbooks [66]. Optical data may vary from one reference to another, leading to different estimations of the Casimir force [67]. Let us emphasize that the problem here is neither due to a lack of precision of the calculations nor to inaccuracies in experiments. The problem is that the calculations and experiments may consider physical samples with different optical properties. This difficulty should be solved by measuring the reflection amplitudes of the mirrors used in the experiment and inserting these informations in the formula giving the predicted Casimir force.

2.4. Temperature correction

The Casimir force between metallic mirrors at nonzero temperature has given rise to contradictory claims which have raised doubts about the theoretical expression of the force. We do not repeat here the discussions which have been devoted to the topic in [20, 26, 27] (see also the recent review [28] and contributions on the topic in the present volume [29]). We only want to stress again that the running controversy can only be solved through an improvement of the knowledge of the reflection amplitudes, particularly at low frequencies. As already discussed, the best manner to do that is to measure these amplitudes on the mirrors used in the experiment.

3. Non-specular scattering

We will now present a more general formalism where the Casimir force and energy are calculated between two objects with non-planar shapes. This formalism is an extension of what has already been presented with the scattering amplitudes now accounting for non-specular reflection. The non-specular case is of course the generic one while specular reflection can only be an idealization. After an introduction to this general formalism, we will discuss applications to the lateral force between corrugated mirrors and we will in particular emphasize deviations from the PFA.

3.1. General scattering formulae

In order to introduce the general formalism, let us first rewrite expression (5) of the Casimir free energy between two parallel plane plates as the sum over modes

$$F = \frac{\partial \mathcal{F}}{\partial L}, \quad \mathcal{F} = i\hbar \int_0^\infty \frac{d\omega}{2\pi} \left(\frac{1}{2} + \bar{n} \right) \ln \det \mathcal{S}, \quad (9)$$

$$\ln \det \mathcal{S} = \text{Tr} \ln \mathcal{S} = \text{Tr} \ln \frac{d^*}{d}, \quad d \equiv 1 - r_1 r_2 e^{2ik_z L}.$$

These equations correspond to the following interpretation [53]: the force F is the change of the free energy \mathcal{F} when the scatterers are being displaced. The free energy \mathcal{F} is described by a

³ Further details on optical data are available; please contact astrid.lambrecht@spectro.jussieu.fr.

storage of vacuum energy due to the scattering process, and is written in terms of the \mathcal{S} -matrix associated with the cavity. As the scattering on stationary objects preserves frequency, this \mathcal{S} -matrix is defined at a given value of ω . As the surfaces are plane and parallel, the scattering also preserves the transverse wavevector \mathbf{k} and polarization p (it only couples modes with opposite values of the longitudinal wavevector). The symbol Tr in (9) refers to a trace over the modes corresponding to different values of \mathbf{k} and p at a fixed frequency. The quantity $\ln \det \mathcal{S}$ can be written in terms of the matrix d which is diagonal on the basis of plane waves, so that equation (9) is effectively equivalent to (5). This ‘scattering formula’ or ‘phaseshift formula’ [53] can equivalently be written as a sum over imaginary frequencies $\omega = i\xi$:

$$\mathcal{F} = \hbar \int_0^\infty \frac{d\xi}{2\pi} (1 + 2\bar{n}) \ln \det d, \quad (10)$$

$$d \equiv 1 - r_1 r_2 \exp(-2\sqrt{\mathbf{k}^2 + \xi^2} L),$$

where d is the denominator of the loop function (2) here written for imaginary frequencies.

As a consequence of this interpretation, it is clear that a more general formula of the Casimir energy can be written in a similar manner for the case of stationary but non-specular scattering [50, 20]. It can be expressed either as a sum over real frequencies, including ordinary and evanescent waves, or as a sum over imaginary frequencies:

$$\mathcal{F} = \hbar \int_0^\infty \frac{d\xi}{2\pi} (1 + 2\bar{n}) \ln \det \mathcal{D}, \quad (11)$$

$$\mathcal{D} \equiv 1 - \mathcal{R}_1 \exp(-\mathcal{K}L) \mathcal{R}_2 \exp(-\mathcal{K}L).$$

The matrices \mathcal{D} , \mathcal{R}_1 and \mathcal{R}_2 are no longer diagonal on the basis of plane waves since they describe non-specular reflection on the two mirrors. The propagation factors contained in \mathcal{K} remain diagonal on the basis of plane waves with their diagonal values written as in (10). Clearly, expression (11) does not depend on the choice of this specific basis. Note that the matrices in (11) do not commute with each other. In particular, the two propagation matrices $\exp(-\mathcal{K}L)$ appearing in \mathcal{D} can be moved through circular permutations in the product but not adjoined to each other.

This equation takes a simpler form at the limit of null temperature (note the change of notation from the free energy \mathcal{F} to the ordinary energy E):

$$F = \frac{dE}{dL}, \quad E = \hbar \int_0^\infty \frac{d\xi}{2\pi} \ln \det \mathcal{D}. \quad (12)$$

Formula (12) has already been used to evaluate the effect of roughness [50] or corrugation [51, 52] of the mirrors. To this aim, it was dealt within a perturbative manner at second order in the roughness or corrugation amplitudes, recalled in the forthcoming paragraphs. It is clear that it has a larger domain of application, not limited to the perturbative regime, as soon as some technique is available for exploiting its general form for specific problems of physical interest. Such a technique has been developed recently by Emig, Graham, Jaffe and Kardar [42], through a multipole expansion well adapted to the treatment of ‘compact’ objects, typically two spheres not too close to each other. The general formula used as the starting point of the expansion is equivalent to our formula (12) with \mathcal{D} given in (11). In particular, the T -matrices in [42] are identified as the non-specular reflection matrices \mathcal{R} of [50]. Meanwhile, the U -matrices in [42] correspond to the propagation matrices $\exp(-\mathcal{K}L)$ of [50], the difference in their explicit expression arising from the fact that they are written on a different basis.

3.2. Scattering formula for the lateral Casimir force

We now come to the discussion of the effect of non-planar geometries and particularly of the deviation from the PFA which could be seen in experiments. As already stated, we thus focus our attention on the lateral Casimir force appearing between corrugated plates. In this case, the deviation of PFA should indeed be visible as a factor in front of the whole effect. This situation is clearly more favorable to the theory/experiment comparison than that met when studying the roughness correction to the normal Casimir force, with this correction being only a small part of the force [50]. Stated differently, the lateral Casimir force could allow for a new test of a prediction of Quantum Electrodynamics, namely the dependence with respect to corrugation wavevector discussed below [51, 52].

Here, we consider two parallel plane mirrors, M1 and M2, with corrugated surfaces described by uniaxial sinusoidal profiles (see figure 1 in [52]):

$$h_1 = a_1 \cos(k_C x), \quad h_2 = a_2 \cos(k_C(x - b)), \quad k_C = \frac{2\pi}{\lambda_C}. \quad (13)$$

The functions $h_1(x, y)$ and $h_2(x, y)$ measure the local height with respect to the mean planes $z_1 = 0$ and $z_2 = L$. They are defined so that h_1 and h_2 have null spatial averages, L thus representing the mean distance between the two surfaces; h_1 and h_2 are both counted as positive when they correspond to separation decreases; λ_C is the corrugation wavelength, k_C is the corresponding wavevector, and b is the spatial mismatch between the corrugation crests.

In the following, we will suppose that the corrugation amplitudes are smaller than the other length scales, namely the corrugation wavelength λ_C , the plasma wavelength λ_P and the interplate distance L :

$$a_1, a_2 \ll \lambda_C, \lambda_P, L. \quad (14)$$

Using the PFA, the Casimir energy is thus obtained by adding the contributions of various surface elements calculated for distributed local distances. Using condition (14) and expanding up to second order in the corrugation amplitudes, we find the lowest-order correction to energy within the PFA,

$$\delta E_{\text{PFA}} = \frac{1}{2} \frac{\partial^2 E_{\text{PP}}}{\partial L^2} \left(\frac{a_1^2 + a_2^2}{2} + a_1 a_2 \cos(k_C b) \right), \quad (15)$$

with E_{PP} the energy calculated between two parallel plane plates. As the energy corrections proportional to a_1^2 and a_2^2 do not depend on the lateral mismatch b , they do not contribute to the lateral force which is simply read as

$$F_{\text{PFA}}^{\text{lat}} = -\frac{\partial \delta E_{\text{PFA}}}{\partial b} = \frac{1}{2} \frac{\partial^2 E_{\text{PP}}}{\partial L^2} k_C a_1 a_2 \sin(k_C b). \quad (16)$$

We will now write the scattering formula for the lateral Casimir force, in a perturbative expansion with respect to the corrugation amplitudes. As in (16), the correction of the Casimir energy will arise at second order in the corrugation amplitudes, with crossed terms of the form $a_1 a_2$ which have the ability to induce lateral forces. The main difference with (15), (16) will be the appearance of a more complicated dependence in the corrugation wavevector k_C .

For this purpose, we expand the non-specular reflection matrix \mathcal{R}_j as the sum $\mathcal{R}_j^{(0)} + \delta \mathcal{R}_j$ of a zeroth-order contribution identified as the specular reflection and of a first-order contribution induced by the reflection on the corrugation [50]. The lowest order modification of the Casimir energy (12) able to produce a lateral force (cross terms $\propto a_1 a_2$) is thus read as

$$\delta E = -\hbar \int_0^\infty \frac{d\xi}{2\pi} \text{Tr} \left(\frac{\exp(-\mathcal{K}L)}{\mathcal{D}^{(0)}} \delta \mathcal{R}_1 \frac{\exp(-\mathcal{K}L)}{\mathcal{D}^{(0)}} \delta \mathcal{R}_2 \right). \quad (17)$$

$\mathcal{D}^{(0)}$ is the matrix \mathcal{D} evaluated at zeroth order in the corrugation. It is diagonal on the basis of plane waves and therefore commutes with \mathcal{K} .

3.3. Explicit results for the plasma model

In order to obtain explicit expressions, it is then necessary to use some microscopic model. To this aim, we study the case of bulk metallic plates described by the plasma dielectric function. The non-specular reflection amplitudes are then calculated in the Rayleigh approximation using techniques which have been developed for treating scattering on rough plates [68, 69]. We want to emphasize that this microscopic model allows one to calculate the lateral Casimir force for arbitrary relative values of the three parameters λ_P , λ_C and L , the corrugation amplitudes remaining the smallest length scale for perturbation theory to hold (see conditions (14)).

This calculation leads to the following expression of the lateral Casimir force:

$$F^{\text{lat}} = -\frac{\partial \delta E}{\partial b}, \quad \delta E = \frac{A}{2} G_C(k_C) a_1 a_2 \cos(k_C b), \quad (18)$$

with the function $G_C(k_C)$ calculated in [52]. It is worth emphasizing that the PFA is recovered in equation (18) as the limiting case $k_C \rightarrow 0$, that is also for long corrugation wavelengths. This follows from a properly formulated ‘proximity force theorem’

$$\lim_{k_C \rightarrow 0} A G_C(k_C) = \frac{d^2 E_{\text{PP}}}{dL^2}. \quad (19)$$

This property is ensured, for any model of the material medium, by the fact that $G_C(k_C \rightarrow 0)$ is given by the specular limit of non-specular reflection amplitudes [52]. This theorem has to be distinguished from the approximation (PFA) which consists in an identification between $G_C(k_C)$ and its limiting value $G_C(0)$. For arbitrary values of k_C , the deviation from the PFA is described by the ratio

$$\rho_C(k_C) = \frac{G_C(k_C)}{G_C(0)}. \quad (20)$$

The variation of this ratio ρ_C with the various parameters has been described in a detailed manner in [51, 52]. Some curves are drawn as examples in figure 1 of [51] with $\lambda_P = 137$ nm chosen to fit the case of gold covered plates. An important feature is that ρ_C is smaller than unity as soon as k_C significantly deviates from 0. For large values of k_C , it even decays exponentially to zero.

4. Concluding remarks

We have studied the lateral Casimir force between two corrugated metallic plates. To this aim, we have used a general scattering formula in a perturbative regime corresponding to corrugation amplitudes smaller than the other length scales L , λ_C and λ_P . The result describes a variety of situations where these three scales have arbitrary relative values. The results known for perfect mirrors [49] are recovered when $\lambda_P \ll \lambda_C, L$. The proximity force approximation (PFA) is recovered at the limit of smooth plates $L, \lambda_P \ll \lambda_C$. A third limiting case has been studied in [52] which corresponds to the opposite case of rugged corrugations $\lambda_C \ll L, \lambda_P$. This case corresponds to evaluations far beyond the PFA regime and is particularly interesting as it constitutes a nontrivial interplay between geometry and the Casimir effect [32]. It is also of great interest for applications to surfaces with structurations at the nanometric scale.

The numerical figures presented in [51, 52] suggest that nontrivial effects of geometry, i.e. effects beyond the PFA, could be observed with dedicated lateral force experiments. Existing experiments by Chen *et al* [47] have used large corrugation amplitudes a_1, a_2 in order to increase the magnitude of the force. As they do not meet the conditions of validity of our

perturbative expansion, it is not possible to compare directly the experimental and theoretical results. Chen *et al* have found their measurements to agree with the PFA to within $\pm 24\%$. Considering smaller amplitudes a_1, a_2 with the same values for the parameters L, λ_C and λ_P , we have obtained a deviation from the PFA of the order of 40%, which means that these parameters do not belong to the domain of validity of the PFA, at least at the perturbative limit.

More work is clearly needed in order to settle this potential concern in the theory–experiment comparison [52, 70, 71]. Progress on this question could be achieved by calculating higher order corrections for metallic mirrors beyond the PFA. These corrections would affect the theoretical predictions, but it seems unlikely that they would compensate exactly the deviation from PFA which has been obtained in the perturbative theory. Progress could alternatively come from experiments with smaller corrugation amplitudes, allowing for a direct comparison with the perturbative theory. A better experimental accuracy would also be very valuable, allowing one to distinguish more easily between alternative predictions. Of course, this program raises serious experimental challenges, given the minuteness of the lateral force effect. But the reward would be remarkable with potentially the first experimental demonstration of a nontrivial interplay between geometry and the Casimir effect.

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